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UNIQUE CONTINUATION FOR $\Delta + v$ AND THE C. FEFFERMAN-PHONG CLASS

SAGUN CHANILLO AND ERIC SAWYER

ABSTRACT. We show that the strong unique continuation property holds for the inequality $|\Delta u| \le |v||u|$, where the potential $v(x)$ satisfies the C. Fefferman-Phong condition in a certain range of p values. We also deal with the situation of $u(x)$ vanishing at infinity. These are all consequences of appropriate Carleman inequalities.

1. INTRODUCTION

Our aim is to prove local and global unique continuation theorems for the differential operator $Q(D) = \Delta + \sum_{j=1}^n a_j \partial/\partial x_j + \gamma$, $a_i, \gamma \in \mathbf{C}$. We assume that for $x \in \Omega \subset \mathbf{R}^n$, $n \ge 3$,

$$|Q(D)u(x)| \le v(x)|u(x)|. \tag{1.1}$$

The problem is to find conditions on $v(x)$ that guarantee that any u satisfying (1.1), and vanishing appropriately at a point, possibly infinity, must vanish identically. In [JK, ST_1 and KRS], unique continuation is proved for v locally in $L^{n/2}$ or locally small in weak $L^{n/2}$, and in addition it is shown that uniqueness can fail for L^p, $p < n/2$. In this paper we consider instead the class of potentials studied by C. Fefferman and D. Phong [F]. We say $v \in F_p$ if

$$\|v\|_{F_p} = \sup_Q |Q|^{2/n} \left(\frac{1}{|Q|} \int_Q |v|^p \right)^{1/p} < \infty.$$

We have $F_q \subset F_p$ if $p < q$ and $F_{n/2} = L^{n/2}$. We will prove unique continuation for v that are locally small in F_p with $p > (n-1)/2$. Note that if $v \in \text{weak}\, L^{n/2}$ then $v \in F_p$ for all $p < n/2$. On the other hand if $v(x) = f(x/|x|)|x|^{-2}$, $f \in L^p(S^{n-1})$, $(n-1)/2 < p < n/2$, then v need not be in weak $L^{n/2}$, but $v \in F_p$. Thus our unique continuation results improve on the corresponding results in [JK, ST_1 and KRS] by permitting potentials in L^p_{loc}, $p > (n-1)/2$, but subject to the growth condition of Fefferman and

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Phong. We do not know if unique continuation holds for v that are locally small in F_p with $1 \le p \le (n-1)/2$. We remark that for $p=1$, the condition F_1 is weaker than the Kato condition,

$$\lim_{\delta \to 0} \sup_{x \in \Omega} \int_{|x-y|<\delta} \frac{|v(y)|}{|x-y|^{n-2}}\, dy = 0,$$

which B. Simon has conjectured is enough to guarantee unique continuation (cf. [Sa]). Part (A) of the following theorem improves the results of [JK] and [St$_1$] while part (B) provides a global analogue. See also [K], where our proof of part (A) appears. Let $H^2(\mathbf{R}^n)$ (respectively $H^2_{\text{loc}}(\mathbf{R}^n)$) denote the Sobolev space of functions whose derivatives up to order two are in $L^2(\mathbf{R}^n)$ (respectively $L^2_{\text{loc}}(\mathbf{R}^n)$).

Theorem (1.2). *Let $u \in H^2_{\text{loc}}(\mathbf{R}^n)$, $n \ge 3$, satisfy $|\Delta u| \le v|u|$.*

(A) *If u vanishes to infinite order at some point $a \in \mathbf{R}^n$, i.e.,*

$$\lim_{R \to 0} R^{-m} \int_{|x-a|<R} |u(x)|^2\, dx = 0 \quad \text{for all } m > 0,$$

and if v is locally small in F_p, i.e.,

$$\overline{\lim_{R \to 0}}\, \|\chi_{\{x:\ |x-y|<R\}} v\|_{F_p} \le \varepsilon(p,n) \quad \text{for all } y \in \mathbf{R}^n,$$

for some $p > (n-1)/2$ and for a sufficiently small positive constant $\varepsilon(p,n)$ depending only on p and n, then u vanishes identically in $\mathbf{R}^n$.

(B) *If $u \in H^2(\mathbf{R}^n)$ vanishes to infinite order at infinity, i.e.,*

$$\lim_{R \to \infty} R^m \int_{|x|>R} |u(x)|^2\, dx = 0 \quad \text{for all } m > 0,$$

and if v satisfies

$$\|\chi_{\{x:\ |x|>R\}} v\|_{F_p} \le \varepsilon(p,n),$$

for some $R > 0$ and $p > (n-1)/2$ and for a sufficiently small positive constant $\varepsilon(p,n)$ depending only on p and n, then $u(x)$ vanishes identically for $|x| > R$.

Our next result is a variant of part (B) of Theorem (1.2) for the differential inequality (1.1) in which u is permitted to merely vanish at infinity in a certain direction, provided it decays very rapidly in the opposite direction. To conserve notation we assume, without loss of generality, that $Q(D) = \Delta + a_n \partial/\partial x_n + \gamma$, a_n real.

Theorem (1.3). *Let $u \in H^2(\mathbf{R}^n)$, $n \ge 3$, satisfy $|Q(D)u| \le v|u|$ and suppose that*

$$\lim_{R \to \infty} \int_{R<|x|<2R} |u(x)|^2 e^{-\lambda x_n}\, dx = 0 \quad \text{for all } \lambda \text{ sufficiently large.}$$

If $v \in F_p$ satisfies $\|\chi_{\{x:\ x_n<t\}} v\|_{F_p} \le \varepsilon(p,n)$ for some real t, where $\varepsilon(p,n)$ is a sufficiently small positive constant depending only on p and n, then $u(x)$ vanishes identically for $x_n < t$.

This theorem is a consequence of a weighted restriction theorem for the Fourier transform, Corollary (2.8), which includes the well-known result of E. Stein and P. Thomas (Corollary (2.9) below).

Finally we mention some further results for dimensions $n = 2$ and 3. For $v \geq 0$ on $\mathbf{R}^n$, let $|||v|||$ denote the least constant C for which the following inequality holds:

$$\left(\int_{\mathbf{R}^n} |f(x)|^2 v(x)\, dx\right)^{1/2} \leq C \left(\int_{\mathbf{R}^n} |\nabla f|^2\right)^{1/2},$$

for all $f \in C_c^\infty(\mathbf{R}^n)$. Equivalently $|||v|||$ is the norm of the embedding I_1: $L^2 \to L^2(v)$, where I_α denotes the fractional integral of order α,

$$I_\alpha f(x) = c_{\alpha,n} \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\, dy, \qquad 0 < \alpha < n.$$

The following theorem improves on all strong unique continuation results for the inequality $|\Delta u| \leq v|u|$ known in dimensions 2 and 3, including Theorem (1.2)(A) and the results of [JK, Sa, and St_1]. Denote by $B(y, R)$ the ball of radius R and centre y.

Theorem (1.4). *Suppose Ω is a bounded, open and connected subset of $\mathbf{R}^n$, $n = 2$ or 3. Let $u \in H^2_{\mathrm{loc}}(\Omega)$ vanish to infinite order at some point in Ω, and suppose that $|\Delta u| \leq v|u|$ in Ω, where v satisfies*

$$\overline{\lim_{R\to 0}}\, |||\chi_{B(y,R)} v||| \leq \varepsilon \quad \text{for all } y \in \Omega,$$

for a sufficiently small positive constant ε. Then u vanishes identically in Ω.

In [KS], it is shown that

$$|||v||| \approx \sup_{\text{cubes } Q} \frac{1}{\int_Q v(x)\, dx} \int_Q \int_Q |x-y|^{2-n} v(x) v(y)\, dx\, dy$$

(with $|x-y|^{2-n}$ replaced by $\log|x-y|$ if $n = 2$) and it is now clear that the Kato condition implies that of Theorem (1.4). In [F], it is shown that $\|v\|_{F_1} \leq |||v||| \leq c_p \|v\|_{F_p}$, $p > 1$. It is an open question whether Theorem (1.4) remains valid for $n \geq 4$.

In dimension $n = 2$, we prove a unique continuation theorem involving a gradient term.

Theorem (1.5). *Suppose Ω is a bounded, open, and connected subset of $\mathbf{R}^2$. Let $u \in H^2_{\mathrm{loc}}(\Omega)$ vanish to infinite order at some point in Ω, and suppose that $|\Delta u| \leq w|\nabla u|$ in Ω, where w^2 is locally small in $F_{p/2}$, i.e.,*

$$\overline{\lim_{R\to 0}} \sup_{\text{cubes } Q} |Q| \left(\frac{1}{|Q|} \int_{Q\cap\{x:\, |x-y|<R\}} w(x)^p\, dx\right)^{2/p} \leq \varepsilon(p)$$

for all $y \in \Omega$, for some $p > 1$ and for a sufficiently small positive constant $\varepsilon(p)$ depending only on p. Then u vanishes identically in Ω.

The differential inequality $|\Delta u| \le w|\nabla u| + v|u|$ is treated for $n \ge 3$ in [BKRS], where it is shown that uniqueness holds if $w \in L^r_{\text{loc}}$ and $v \in L^s_{\text{loc}}$, $r = (3n-2)/2$, $s > n/2$. By Theorem (1.5), strong uniqueness holds for $|\nabla u| \le w|\nabla u|$ if $w \in L^2_{\text{loc}}$, $n = 2$. It is an open question whether uniqueness holds for $|\Delta u| \le w|\nabla u|$ if $w \in L^n_{\text{loc}}$, $n \ge 3$. The proofs of Theorems (1.2) and (1.3) are in §3 and §2 respectively and the proofs of Theorems (1.4) and (1.5) are in §4.

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We close this section by recording a pair of elementary inequalities that will be used in the next two sections.

Lemma (1.6). *Let f be in $H^2_{\text{loc}}(\mathbf{R}^n)$, $a > 0$ and $\lambda \in \mathbf{R}$. Then*

$$\text{(A)}\quad \begin{aligned}\int_{|x|\le a} |\nabla f(x)|^2\,dx \le C\left(\int_{|x|\le 2a} |f(x)|^2\,dx\right)^{1/2}\left(\int_{|x|\le 2a} |\Delta f(x)|^2\,dx\right)^{1/2}\\ + Ca^{-2}\int_{|x|\le 2a}|f(x)|^2\,dx,\end{aligned}$$

$$\text{(B)}\quad \begin{aligned}\int_{a\le|x|\le 2a} |\nabla f(x)|^2 e^{-\lambda x_n}\,dx \le C\left(\int_{a/2\le|x|\le 4a} |f(x)|^2 e^{-2\lambda x_n}\,dx\right)^{1/2}\\ \times\left(\int_{a/2\le|x|\le 4a}|\Delta f(x)|^2\,dx\right)^{1/2}\\ + C(a^{-2}+\lambda^2)\int_{a/2\le|x|\le 4a}|f(x)|^2 e^{-\lambda x_n}\,dx.\end{aligned}$$

The constant C is independent of f, a, and λ.

Proof. (A) It suffices to consider the case $a = 1$. Let $\theta \in C^\infty_c(B(0,2))$ satisfy $\theta = 1$ on $B(0,1)$. Then

$$\int |\theta\nabla f|^2 \le 2\int |f\nabla\theta|^2 + 2\int|\nabla(\theta f)|^2 = 2\int|f\nabla\theta|^2 - 2\int \theta f\Delta(\theta f).$$

Now use

$$\begin{aligned}|\theta f\Delta(\theta f)| &\le |\theta f||\theta\Delta f| + 2|f\nabla\theta||\theta\nabla f| + \theta|\Delta\theta||f|^2\\ &\le |\theta f||\theta\Delta f| + 16|\nabla\theta|^2|f|^2 + \tfrac{1}{4}|\theta\nabla f|^2 + \theta|\Delta\theta||f|^2\end{aligned}$$

to get

$$\text{(1.7)}\quad \begin{aligned}\int|\theta\nabla f|^2 \le C\left[\int(|\nabla\theta|^2+\theta|\Delta\theta|)|f|^2 + \left(\int|\theta f|^2\right)^{1/2}\left(\int|\theta\Delta f|^2\right)^{1/2}\right]\\ + \frac{1}{2}\int|\theta\nabla f|^2,\end{aligned}$$

and then subtract the last summand from both sides.

(B) Take $a = 1$ and let $\varphi \in C_c^\infty(B(0,4)\backslash B(0,\frac{1}{2}))$ satisfy $\varphi = 1$ on $B(0,2)\backslash B(0,1)$. Set $\theta = e^{-(\lambda/2)x_n}\varphi$. Arguing as in the proof of part (A) yields the following variant of (1.7),

$$\int |\theta\nabla f|^2 \leq C\left[\int (|\nabla\theta|^2 + \theta|\Delta\theta|)|f|^2 + \left(\int |\theta^2 f|^2\right)^{1/2}\left(\int_{1/2\leq|x|\leq 4} |\Delta f|^2\right)^{1/2}\right] + \frac{1}{2}\int |\theta\nabla f|^2.$$

Now use $|\theta| \leq Ce^{-(\lambda/2)x_n}$, $|\nabla\theta| \leq C(1+|\lambda|)e^{-(\lambda/2)x_n}$ and $|\Delta\theta| \leq C(1+\lambda^2)e^{-(\lambda/2)x_n}$.

2. Proof of Theorem (1.3)

We follow the argument of [KRS] beginning with a weighted L^2 version of their Carleman inequality.

Lemma (2.1). *Let $Q(D) = \Delta + a_n\partial/\partial x_n + \gamma$ where $n \geq 3$, $a_n \in \mathbf{R}$ and $\gamma \in \mathbf{C}$. Suppose $v \geq 0$ is in F_p with $p > (n-1)/2$. Then for $f \in C_0^\infty(\mathbf{R}^n)$ and λ real,*

$$\int_{\mathbf{R}^n} |f|^2 e^{-2\lambda x_n} v\, dx \leq C\int_{\mathbf{R}^n} |Q(D)f|^2 e^{-2\lambda x_n} v^{-1}\, dx$$

with C independent of f, λ, a_n, and γ.

We begin with a weighted restriction theorem which will substitute for Lemma 2.1(a) of [KRS]. Let J_β denote the Bessel function of order β.

Lemma (2.2). *Let $n \geq 2$ and $2n/(n+1) \leq \delta \leq (n+1)/2$. Set $p_0 = (n-1)/2(\delta-1)$ and let $p > p_0$ if $\delta > 2n/(n+1)$ and $p = p_0$ if $\delta = 2n/(n+1)$. Suppose $v \geq 0$ and that for all cubes $Q \subset \mathbf{R}^n$,*

$$|Q|^{\delta/n}\left(\frac{1}{|Q|}\int_Q v^p\, dx\right)^{1/p} \leq C.$$

Then

$$\int_{\mathbf{R}^n} |f * \widehat{d\sigma}|^2 v\, dx \leq C\int_{\mathbf{R}^n} |f|^2 v^{-1}\, dx,$$

where

$$\widehat{d\sigma}(x) = \int_{S^{n-1}} e^{i\langle x,\xi\rangle}\, d\sigma(\xi) = c_n J_{(n-2)/2}(|x|)|x|^{-(n-2)/2}.$$

Proof. Before proving the lemma we note that the restriction $\delta \leq (n+1)/2$ is to ensure that $p_0 \geq 1$. We consider first the case $\delta > 2n/(n+1)$. Let $v^*(x) = M(v^q)^{1/q}(x)$, where $p > q > p_0$ and where M is the Hardy-Littlewood maximal function. Define

$$T_z f(x) = [K_z * (fv^{*z/2})(x)]v^*(x)^{z/2}, \quad \text{where } K_z(x) = \frac{J_{n/2-z}(|x|)}{|x|^{n/2-z}}.$$

We will apply Stein's theorem on complex interpolation of operators [St_2]. First, for $\operatorname{Re} z = 0$, we have by [T],

$$\int_{\mathbf{R}^n} |T_z f|^2 \, dx \le C_z \int_{\mathbf{R}^n} |f|^2 \, dx . \tag{2.3}$$

Let $\operatorname{Re} z = p_0$ and $\alpha = (n-1)/2 + p_0$. Since $|K_z(x)| \le C_z |x|^{-(n+1)/2+p_0}$, we have $\alpha < n$ and

$$|T_z f(x)| \le C_z I_\alpha(|f| v^{*p_0/2}) v^{*p_0/2}(x) . \tag{2.4}$$

We have used the estimate $|J_\beta(|x|)| \le C_\beta |x|^{-1/2}$ for $\operatorname{Re}\beta \ge -1/2$, which applies here as $p_0 \le (n+1)/2$ for $\delta \ge 2n/(n+1)$. Thus, for $\operatorname{Re} z = p_0$, $\alpha = (n-1)/2 + p_0$, from (2.4) we get,

$$\int_{\mathbf{R}^n} |T_z f|^2 \, dx \le C_z \int_{\mathbf{R}^n} [I_\alpha(|f| v^{*p_0/2})]^2 v^{*p_0} \, dx .$$

By [CR], $v^{*p_0} \in A_1$, and $v^{*-p_0} \in A_2$ and so by [CW],

$$\int_{\mathbf{R}^n} [I_\alpha(|f| v^{*p_0/2})]^2 v^{*p_0} \le C \int_{\mathbf{R}^n} |f v^{*p_0/2}|^2 (v^*)^{-p_0} = C \int_{\mathbf{R}^n} |f|^2 \, dx \tag{2.5}$$

provided, for $r > 1$,

$$|Q|^{2\alpha/n} \left(\frac{1}{|Q|} \int_Q v^{*rp_0} \right)^{1/r} \le \frac{1}{|Q|} \int_Q v^{*-p_0} . \tag{2.6}$$

Because $v^{*-p_0} \in A_2$, (2.6) is equivalent to

$$|Q|^{2\alpha/n} \left(\frac{1}{|Q|} \int_Q v^{*rp_0} \right)^{2/r} \le C .$$

Taking square roots and setting $rp_0 = p$ and noting $\alpha/p_0 = \delta$, we get that (2.5) follows, provided

$$|Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q v^{*p} \right)^{1/p} \le C, \qquad p_0 < q < p . \tag{2.7}$$

To see (2.7), we use the argument of [F, proof of Lemma C]. Set $v_1 = v\chi_{2Q}$, $v_2 = v - v_1$. Clearly, the left side of (2.7) is bounded by

$$|Q|^{\delta/n} \left(\frac{1}{|Q|} \int_Q v_1^{*p} \right)^{1/p} + |Q|^{\delta/n} \left(\frac{1}{|Q|} \int_Q v_2^{*p} \right)^{1/p} .$$

By the definition of v_2, for $x \in Q$,

$$v_2^*(x) \le \left(\frac{1}{|B|} \int_B |v_2|^q \right)^{1/q}$$

for some cube $B \supset Q$. Thus because $p > q$,

$$|Q|^{\delta/n} \left(\frac{1}{|Q|} \int_Q v_2^{*p} \right)^{1/p} \le |B|^{\delta/n} \left(\frac{1}{|B|} \int_B v_2^q \right)^{1/q} \le |B|^{\delta/n} \left(\frac{1}{|B|} \int_B v^q \right)^{1/q} \le C .$$

By the maximal theorem,

$$\begin{aligned}|Q|^{\delta/n}\left(\frac{1}{|Q|}\int_Q v_1^{*p}\right)^{1/p} &\le |Q|^{\delta/n}\left(\frac{1}{|Q|}\int_{\mathbf{R}^n} v_1^{*p}\right)^{1/p}\\ &\le |Q|^{\delta/n}\left(\frac{1}{|Q|}\int_{\mathbf{R}^n} v_1^{p}\right)^{1/p}\\ &\le |Q|^{\delta/n}\left(\frac{1}{|Q|}\int_{2Q} v^{p}\right)^{1/p} \le C.\end{aligned}$$

Thus (2.7) is proved. Therefore (2.3) holds for $\operatorname{Re} z = p_0$, at least in the case $\delta > 2n/(n+1)$. In the case $\delta = 2n/(n+1)$, $p = p_0$, set

$$T_z f(x) = [K_z * (f v^{z/2})(x)] v^{z/2}(x).$$

Inequality (2.3) persists for $\operatorname{Re} z = 0$. Since $|K_z(x)| \le C_z$ for $\operatorname{Re} z = p_0 = (n+1)/2$,

$$|T_z f(x)|^2 \le \left[\int |f|^2\, dx\right]\left[\int v^{(n+1)/2}\, dx\right] v(x)^{(n+1)/2}$$

by Hölder's inequality and so (2.3) holds for $\operatorname{Re} z = p_0 = (n+1)/2$ if

$$\int v^{(n+1)/2}\, dx < \infty$$

as required. By interpolation, (2.3) holds for $\operatorname{Re} z = 1$. Thus

$$\begin{aligned}\int_{\mathbf{R}^n} |f * \widehat{d\sigma}|^2 v \le \int_{\mathbf{R}^n} |f * \widehat{d\sigma}|^2 v^* &= \int_{\mathbf{R}^n} |T_1(f v^{*-1/2})|^2\, dx\\ &\le C\int_{\mathbf{R}^n} |f|^2 v^{*-1}\, dx \le C\int_{\mathbf{R}^n} |f|^2 v^{-1}\, dx,\end{aligned}$$

and the lemma is proved.

We now obtain the corollaries that follow.

Corollary (2.8). *Let v satisfy the hypothesis of Lemma* (2.2). *Then,*

$$\int_{S^{n-1}} |\hat f|^2\, d\sigma \le C\int_{\mathbf{R}^n} |f|^2 v^{-1}\, dx.$$

Proof. We use P. Tomas's duality argument.

$$\int_{S^{n-1}} |\hat f|^2\, d\sigma = \int_{\mathbf{R}^n} f(\overline{f} * \widehat{d\sigma}) \le \|f\|_{L^2_{v^{-1}}} \|f * \widehat{d\sigma}\|_{L^2_v}.$$

By Lemma (2.2),

$$\|f * \widehat{d\sigma}\|_{L^2_v} \le C\|f\|_{L^2_{v^{-1}}},$$

and the corollary now follows.

Corollary (2.9) [T]. *Let $n \ge 2$ and $1 \le p \le 2(n+1)/(n+3)$. Then*

$$\left(\int_{S^{n-1}} |\hat f|^2\, d\sigma\right)^{1/2} \le C_p\left(\int_{\mathbf{R}^n} |f|^p\right)^{1/p}.$$

Proof. Set $v(x) = |f|^{2-p}$ and assume $\|f\|_p = 1$. Note that

$$\int_{\mathbf{R}^n} |f|^p\,dx = \int_{\mathbf{R}^n} |f|^2 v^{-1}\,dx.$$

If $v \in L^{n/\delta}$, then the hypothesis of Corollary (2.8) will be fulfilled and we would have

$$\int_{S^{n-1}} |\hat{f}|^2\,d\sigma \leq C\int_{\mathbf{R}^n} |f|^2 v^{-1} = C\int_{\mathbf{R}^n} |f|^p = C.$$

For $v \in L^{n/\delta}(\mathbf{R}^n)$, we need $|f|^{(2-p)n/\delta} = |f|^p$, or $(2-p)n/\delta = p$. This forces $p = 2n/(n+\delta)$. But for $\delta \geq 2n/(n+1)$, $p \leq 2(n+1)/(n+3)$ and we are done.

The result of Corollary (2.9) was obtained earlier by E. Stein and P. Tomas [T]. We remark that the sharp result with $p = 2(n+1)/(n+3)$ in Corollary (2.9) is equivalent to the case $\delta = 2n/(n+1)$ of Lemma (2.2) since Holder's inequality shows that

$$\left[\int |f|^p\,dx\right]^{1/p} \leq \left[\int |f|^2 v^{-1}\,dx\right]^{1/2}\left[\int v^{(n+1)/2}\,dx\right]^{1/(n+1)}$$

if $p = 2(n+1)/(n+3)$. However, we shall be interested in the special case $\delta = 2$ of Lemma (2.2), which neither implies nor is implied by Corollary (2.9). Our next result serves as a substitute for Lemma 2.1(b) of [KRS].

Lemma (2.10). *Let* $v \in F_p$, $p > (n-1)/2$. *Suppose* $\operatorname{Im} z \neq 0$ *and* $\operatorname{Re} z < 0$ *and let*

$$\widehat{Tf}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 + z}.$$

Then,

$$\int_{\mathbf{R}^n} |Tf|^2 v\,dx \leq C\int_{\mathbf{R}^n} |f|^2 v^{-1}\,dx$$

with C *independent of* z.

Proof. We mimic the proof of Lemma (2.2) in the special case $\delta = 2$. Let $v^*(x)$ be as in Lemma (2.2) with $p_0 = (n-1)/2$. Clearly it is enough to show

$$\int_{\mathbf{R}^n} |Tf|^2 v^*\,dx \leq c\int_{\mathbf{R}^n} |f|^2 v^{*-1}\,dx.$$

Define

$$\widehat{T_\mu f}(\xi) = \frac{\hat{f}(\xi)}{(|\xi|^2 + z)^\mu}, \qquad \operatorname{Im} z \neq 0, \operatorname{Re} z < 0,$$

and let $G_\mu f(x) = [T_\mu(fv^{*\mu/2})]v^{*\mu/2}(x)$.

Note that if $\operatorname{Re}\mu = 0$ and $\operatorname{Im} z \neq 0$ we have $\|G_\mu f\|_2 \leq \|f\|_2$. If $\operatorname{Re}\mu = (n-1)/2$, the kernel $R_\mu(x)$ of T_μ satisfies

$$|R_\mu(x)| \leq C|x|^{-1}. \tag{2.11}$$

This is seen by examining the formula for $R_\mu(x)$ in [KRS and GS]. In fact,

$$(2.12)\qquad R_\mu(x) = \frac{e^{\mu^2}2^{-\mu+1}}{(2\pi)^{n/2}\Gamma(\mu)\Gamma(n/2-\mu)}\left[\frac{z}{|x|^2}\right]^{(1/2)(n/2-\mu)} K_{n/2-\mu}\left(\sqrt{z|x|^2}\right)$$

where K_δ is the Bessel kernel of the third kind. It is known that

$$(2.13)\qquad |K_\delta(w)| \le \begin{cases} c|w|^{-|\mathrm{Re}\,\delta|}, & \text{if } |w| \le 1,\ \mathrm{Re}\,w > 0, \\ c|w|^{-1/2}, & \text{if } |w| \ge 1,\ \mathrm{Re}\,w > 0. \end{cases}$$

If $\delta = n/2 - \mu$, $\mathrm{Re}\,\mu = (n-1)/2$ and $w = \sqrt{z|x|^2}$, we see $\mathrm{Re}\,\delta = 1/2$ and $\mathrm{Re}\,w > 0$ as $\mathrm{Re}\,z < 0$. Therefore we may estimate (2.12) by (2.13) to get (2.11). Thus if $\mathrm{Re}\,\mu = (n-1)/2$,

$$|G_\mu f(x)| \le I_{n-1}(|f|v^{*(n-1)/4})v^{*(n-1)/4},$$

and so

$$\int_{\mathbf{R}^n} |G_\mu f|^2\,dx \le \int_{\mathbf{R}^n} [I_{n-1}(|f|v^{*(n-1)/4})]^2 v^{*(n-1)/2}\,dx.$$

But by the arguments of Lemma (2.2), the integral on the right side above is at most $\int_{\mathbf{R}^n} |f|^2\,dx$ because $v^* = (Mv^q)^{1/q} \in F_p$ for $1 < q < p$ (see the argument following (2.7) or the proof of Lemma C, p. 153, in [F]). The lemma now follows using complex interpolation, and then taking $\mu \to 1$.

The next lemma is needed to adapt the Littlewood-Paley arguments of [KRS] to the weighted setting. Note the restriction to $p > (n-1)/2$.

Lemma (2.14). *Let $v \in F_p$, $p > (n-1)/2$. For $s \ge 1$, define the one-dimensional maximal function $v_*(x)$ by*

$$v_*(x) = \sup_\mu \left(\frac{1}{2\mu}\int_{x_n-\mu}^{x_n+\mu} v(x_1, x_2, \dots, x_{n-1}, t)^s\,dt\right)^{1/s}.$$

Then, for $p > \max(s, (n-1)/2)$

$$\sup_Q |Q|^{2/n}\left(\frac{1}{|Q|}\int_Q v_*^p\right)^{1/p} \le C \sup_Q |Q|^{2/n}\left(\frac{1}{|Q|}\int_Q v^p\right)^{1/p}.$$

That is, $v_* \in F_p$.

Proof. Fix a cube Q with center at z and edge length δ. Define the rectangle R_k, $k \ge 1$, such that $(y_1, y_2, \dots, y_{n-1}, y_n) \in R_k$ if $|y_j - z_j| \le 2\delta$, $j = 1, 2, \dots, n-1$, but $2^k\delta < |y_n - z_n| \le 2^{k+1}\delta$. Let $R_0 = 4Q$. Let $v^{(k)} = v\chi_{R_k}$. Then, $v(x) \le \sum_{k=0}^\infty v^{(k)}(x) + \varphi(x)$, where $\varphi(x)$ is supported on $\mathbf{R}^n \setminus \bigcup_{k=0}^\infty R_k$. Thus,

$$(2.15)\qquad |Q|^{2/n}\left(\frac{1}{|Q|}\int_Q v_*^p\right)^{1/p} \le \sum_{k=0}^\infty |Q|^{2/n}\left(\left(\frac{1}{|Q|}\int_Q v_*^{(k)p}\right)^{1/p} + \left(\frac{1}{|Q|}\int_Q \varphi_*^p\right)^{1/p}\right).$$

But for $x \in Q$, $\varphi_*(x) = 0$. By the maximal theorem when $k = 0$,

$$|Q|^{2/n}\left(\frac{1}{|Q|}\int_Q v_*^{(0)p}\right)^{1/p} \leq C|Q|^{2/n}\left(\frac{1}{|Q|}\int_{4Q} v^p\right)^{1/p} \leq c.$$

If $k \neq 0$, note that for $(x_1, \ldots, x_n) \in Q$,

$$v_*^{(k)}(x)^p \leq \frac{C}{2^k\delta}\int_{2^k\delta<|z_n-t|\leq 2^{k+1}\delta} v(x_1, \ldots, x_{n-1}, t)^p\,dt.$$

Thus for $k \neq 0$

$$\frac{1}{|Q|}\int_Q v_*^{(k)p}\,dx \leq \frac{C}{2^k\delta^n}\int_{R_k} v^p\,dx.$$

Now there exists a cube Q_k such that $Q_k \supset R_k$ and $|Q_k| \sim (2^k\delta)^n$. Thus

$$\frac{c}{\delta^{n-1}(2^k\delta)}\int_{R^k} v^p\,dx \leq C2^{(n-1)k}\frac{1}{|Q_k|}\int_{Q_k} v^p\,dx.$$

Moreover $|Q|^{2/n} = 2^{-2k}|Q_k|^{2/n}$. Using this and the fact that for all k

$$\left(\frac{1}{|Q|}\int_Q v_*^{(k)p}\right)^{1/p} \leq C2^{(n-1)k/p}\left(\frac{1}{|Q_k|}\int_{Q_k} v^p\,dx\right)^{1/p},$$

the right side of (2.15) is at most $C\sum_{k=0}^{\infty} 2^{(-2+(n-1)/p)k}$. This converges provided $p > (n-1)/2$.

Proof of Lemma (2.1). We follow the argument of [KRS]. Conjugating the operator $Q(D)$ by $e^{-\lambda x_n}$ we see that, if $\gamma = \alpha + i\mu$, we have

$$\begin{aligned}(e^{-\lambda x_n}Q(D)e^{\lambda x_n})\hat{\ }(\xi) &= -(|\xi|^2 - \lambda^2 - a_n\lambda - i\xi_n(2\lambda - a_n) - \gamma)\\ &= -\left(|\xi|^2 - \lambda^2 - a_n\lambda - \alpha - 2i\lambda\left(\xi_n - \frac{a_n\lambda^{-1}}{2} + \frac{\mu\lambda^{-1}}{2}\right)\right)\\ &= -(|\xi|^2 - \lambda^2 - a_n\lambda - \alpha - 2i\lambda(\xi_n + \delta)),\end{aligned}$$

where $\delta = -a_n\lambda^{-1}/2 + \mu\lambda^{-1}/2$. Thus it will be enough to show,

$$\int_{\mathbf{R}^n} |Sf|^2 v\,dx \leq c\int_{\mathbf{R}^n} |f|^2 v^{-1}\,dx, \tag{2.16}$$

where

$$\widehat{Sf}(\xi) = \hat{f}(\xi)[|\xi|^2 - \lambda^2 - a_n\lambda - \alpha - 2i\lambda(\xi_n + \delta)]^{-1} = \hat{f}(\xi)m(\xi).$$

Let $\chi_k(\xi_n) = \chi(2^{-k}(\xi_n + \delta))$ be a smooth Littlewood-Paley-Stein decomposition of $\mathbf{R}$. Let $\widehat{R_k f}(\xi) = \chi_k(\xi_n)m(\xi)\hat{f}(\xi)$. Define v_* as in Lemma (2.14) for some s with $1 < s < p$. Then $v_*(x) \in A_1(\mathbf{R}, dx_n)$. Thus by applying [Ku] to the x_n variable,

$$\int_{\mathbf{R}^n} |Sf|^2 v\,dx \leq \int_{\mathbf{R}^n} |Sf|^2 v_*\,dx \leq \sum_k \int_{\mathbf{R}^n} |R_k f|^2 v_*\,dx. \tag{2.17}$$

Let $\hat{f}_k(\xi) = \chi_k(\xi_n)\hat{f}(\xi)$. We claim

$$\int_{\mathbf{R}^n} |R_k f|^2 v_* \, dx \le C \int_{\mathbf{R}^n} |f_k|^2 v_*^{-1} \, dx \tag{2.18}$$

with a constant c independent of k and λ. Now $v_*^{-1} \in A_2(\mathbf{R}, dx_n)$. Assuming (2.18), the weighted Littlewood-Paley-Stein theorem of [Ku] shows that

$$\begin{aligned}\sum_k \int_{\mathbf{R}^n} |R_k f|^2 v_* \, dx &\le C \sum_k \int_{\mathbf{R}^n} |f_k|^2 v_*^{-1} \, dx \\ &\le C \int_{\mathbf{R}^n} |f|^2 v_*^{-1} \, dx \le c \int_{\mathbf{R}^n} |f|^2 v^{-1} \, dx .\end{aligned}$$

Combining this with (2.17) yields (2.16).

We now prove (2.18). We have $R_k = A_k + B_k$ with

$$\widehat{A_k f}(\xi) = \frac{\chi_k(\xi_n)}{|\xi|^2 - \lambda^2 - a_n\lambda - \alpha - i2^{k+1}\lambda} \hat{f}(\xi),$$

$$\begin{aligned}\widehat{B_k f}(\xi) = \chi_k(\xi_n) \Bigg[&\frac{1}{|\xi|^2 - \lambda^2 - \alpha_n\lambda - \alpha - 2i\lambda(\xi_n + \delta)} \\ &- \frac{1}{|\xi|^2 - \lambda^2 - a_n\lambda - \alpha - i2^{k+1}\lambda} \Bigg] \hat{f}(\xi) .\end{aligned}$$

Observe that

$$\widehat{A_k f}(\xi) = \frac{1}{|\xi|^2 + z} \chi_k(\xi_n)\hat{f}(\xi)$$

with $\operatorname{Re} z < 0$, $\operatorname{Im} z \neq 0$.

By Lemma (2.14), $v_* \in F_p$, $p > (n-1)/2$. If T is as in Lemma (2.10), we have

$$\int_{\mathbf{R}^n} |A_k f|^2 v_* = c \int_{\mathbf{R}^n} |T f_k|^2 v_* \le c \int_{\mathbf{R}^n} |f_k|^2 v_*^{-1} .$$

As in [KRS], we write $B_k f(x)$ in polar coordinates with $|\xi| = r$,

$$\begin{aligned}B_k f(x) = \int_0^\infty \int_{S^{n-1}} &e^{i\langle r\sigma, x\rangle} \varphi_2(r)\chi_k(r\sigma_n)\hat{f}(r\sigma) \\ &\times \Bigg[\frac{1}{r^2 - \lambda^2 - a_n\lambda - \alpha - 2i\lambda(r\sigma_n + \delta)} \\ &- \frac{1}{r^2 - \lambda^2 - a_n\lambda - \alpha - i2^{k+1}\lambda} \Bigg] r^{n-1} \, d\sigma \, dr .\end{aligned}$$

Thus,

$$|B_k f(x)| \le 2 \int_0^\infty \left| \int_{S^{n-1}} m_{k,r}(r\sigma)\hat{f}_k(r\sigma) e^{i\langle\sigma, rx\rangle} \, d\sigma \right| r^{n-1} \, dr$$

with

$$m_{k,r}(\xi) = \frac{\lambda(\xi_n + \delta - 2^k)\psi_k(\xi_n)}{(r^2 - \lambda^2 - a_n\lambda - \alpha - 2i\lambda(\xi_n + \delta))(r^2 - \lambda^2 - a_n\lambda - \alpha - i2^{k+1}\lambda)},$$

where $\psi_k \equiv 1$ on $\operatorname{supp}\chi_k$ and ψ_k supported near $|\xi_n+\delta| \approx 2^k$. Thus,

$$|B_k f(x)| \le C\int_0^\infty |\delta_r(T_{m_{k,r}} f_k) * \widehat{d\sigma}(rx)| r^{n-1}\,dr,$$

where $\delta_r h(x) = (1/r^n)h(x/r)$ and $T_{m_{k,r}}$ denotes the multiplier with symbol $m_{k,r}$. By Minkowski's inequality,

$$\left(\int_{\mathbf{R}^n} |B_k f|^2 v * dx\right)^{1/2} \le C\int_0^\infty \left(\int_{\mathbf{R}^n} |\delta_r(T_{m_{k,r}} f_k) * \widehat{d\sigma}(rx)|^2 v_*(x)\right)^{1/2} r^{n-1}\,dr.$$

Changing variables by setting $rx = y$, the last integral is

$$C\int_0^\infty \left(\int_{\mathbf{R}^n} |\delta_r(T_{m_{k,r}} f_k) * \widehat{d\sigma}(y)|^2 v_*\left(\frac{y}{r}\right)\,dy\right)^{1/2} r^{n/2-1}\,dr.$$

Let $w(y) = (1/r^2)v_*(y/r)$. Then by Lemma (2.14) and a rescaling,

$$\left(\frac{1}{|Q|}\int_Q w^p\right)^{1/p} |Q|^{2/n} \le C.$$

Thus by Lemma (2.2) the integral above is dominated by

$$C\int_0^\infty \left(\int_{\mathbf{R}^n} r^{-2n}\left|T_{m_{k,r}} f_k\left(\frac{x}{r}\right)\right|^2 v_*^{-1}\left(\frac{x}{r}\right)\,dx\right)^{1/2} r^{n/2+1}\,dr,$$

and by changing variables again this integral is at most

$$\int_0^\infty \left(\int_{\mathbf{R}^n} |T_{m_{k,r}} f_k(x)|^2 v_*^{-1}(x)\,dx\right)^{1/2} r\,dr. \tag{2.19}$$

Now, one easily checks that,

$$|m_{k,r}(\xi_n)| \le \frac{C\lambda 2^k}{(r^2-\lambda^2-a_n\lambda-\alpha)^2 + 2^{2k}\lambda^2}$$

and

$$|m'_{k,r}(\xi_n)| \le \frac{C\lambda}{(r^2-\lambda^2-a_n\lambda-\alpha)^2 + 2^{2k}\lambda^2}.$$

Thus $m_{k,r}(\xi_n)$ satisfies the conditions of the Marcinkiewicz multiplier theorem. Since $v_*^{-1} \in A_2(\mathbf{R}, dx_n)$, by [Ku], (2.19) above is at most

$$\begin{aligned} &C\left(\int_{\mathbf{R}^n} |f_k|^2 v_*^{-1}\,dx\right)^{1/2} \int_0^\infty \frac{\lambda 2^k r}{(r^2-\lambda^2-a_n\lambda-\alpha)^2 + 2^{2k}\lambda^2}\,dr \\ &= C\left(\int_{\mathbf{R}^n} |f_k|^2 v_*^{-1}\,dx\right)^{1/2} \int_{-\infty}^\infty \frac{dt}{t^2+1}, \qquad 2^k\lambda t = r^2-\lambda^2-a_n\lambda-\alpha, \\ &= \pi C\left(\int_{\mathbf{R}^n} |f_k|^2 v_*^{-1}\,dx\right)^{1/2}. \end{aligned}$$

This proves (2.18) and, as mentioned earlier, completes the proof of Lemma (2.1).

Proof of Theorem (1.3). Since $|Q(D)u| \le v|u| \le (v + \varepsilon/|x|^2)|u|$, and $v + \varepsilon/|x|^2$ also satisfies the hypothesis of Theorem (1.2), we may assume $v(x) \ge \varepsilon |x|^{-2}$. Select $\eta \in C_0^\infty(B(0,2))$ with $\eta \equiv 1$ on $B(0,1)$, set $\eta_R(x) = \eta(x/R)$ and let $S_t = \{(x_1, \dots, x_n), x_n < t\}$. By hypothesis, $\|\chi_{S_t} v\|_{F_p} \le \varepsilon(p, n)$ and thus the constant C in the conclusion of Lemma (2.1) may be taken to be at most $c_0 < 1$ if v is replaced by $\chi_{S_t} v$. Thus with $\lambda > 0$ in Lemma (2.1),

(2.20)

$$\begin{aligned}\int_{S_t} |\eta_R u|^2 e^{-\lambda x_n} v\, dx &\le c_0 \int_{\mathbf{R}^n_+} |Q(D)(\eta_R u)|^2 e^{-\lambda x_n} v^{-1}\, dx \\ &\le c_0 \left\{ \int_{\mathbf{R}^n} |\eta_R Q(D) u|^2 e^{-\lambda x_n} v^{-1}\, dx + C \int_{R \le |x| \le 2R} (|u|^2 + |\nabla u|^2) e^{-\lambda x_n}\, dx \right\}\end{aligned}$$

since $v^{-1} \le C|x|^2$, $|\nabla \eta_R| \le CR^{-1}$ and $|\Delta \eta_R| \le CR^{-2}$. For the first term on the right side of (2.20), the differential inequality yields

(2.21)

$$\int_{\mathbf{R}^n} |\eta_R Q(D) u|^2 e^{-\lambda x_n} v^{-1}\, dx \le \int_{\mathbf{R}^n \setminus S_t} |\eta_R u|^2 e^{-\lambda x_n} v\, dx + \int_{S_t} |\eta_R u|^2 e^{-\lambda x_n} v\, dx .$$

Since $\chi_{S_t} v \in F_p$, $p > 1$, we have by [F]

$$\int_{S_t} |\eta_R u|^2 e^{-\lambda x_n} v \le e^{2|\lambda| R} \int_{S_t} |\eta_R u|^2 v \le C e^{2|\lambda| R} \int_{\mathbf{R}^n} |\nabla(\eta_R u)|^2 ,$$

and thus the second integral on the right side of (2.21) is finite. So from (2.20) we conclude

$$\begin{aligned}\int_{S_t} |\eta_R u|^2 e^{-\lambda x_n} v\, dx &\le C \int_{\mathbf{R}^n \setminus S_t} |\eta_R u|^2 e^{-\lambda x_n} v\, dx \\ &\quad + C \int_{R \le |x| \le 2R} (|u|^2 + |\nabla u|^2) e^{-\lambda x_n}\, dx .\end{aligned}$$

Multiplying by $e^{\lambda t}$ and using [F] again, we have

$$\begin{aligned}\int_{S_t \cap B_R} |u|^2 e^{\lambda(t - x_n)} v\, dx &\le C \int_{\mathbf{R}^n} |\eta_R u|^2 v + C e^{\lambda t} \int_{R \le |x| \le 2R} (|u|^2 + |\nabla u|^2) e^{-\lambda x_n}\, dx \\ &\le C \int_{\mathbf{R}^n} |\nabla(\eta_R u)|^2 + C e^{\lambda t} \int_{R \le |x| \le 2R} (|u|^2 + |\nabla u|^2) e^{-\lambda x_n}\, dx .\end{aligned}$$

Letting $R \to \infty$ and using Lemma (1.6)(B) together with the hypothesis on u, we get

$$\begin{aligned}\int_{S_t} |u|^2 e^{\lambda(t - x_n)} v\, dx &\le C \lim_{R \to \infty} \int_{\mathbf{R}^n} (|\nabla \eta_R|^2 |u|^2 + \eta_R^2 |\nabla u|^2)\, dx \\ &\le C \int_{\mathbf{R}^n} |\nabla u|^2\, dx < \infty .\end{aligned}$$

Letting $\lambda \to \infty$, we see that $u \equiv 0$ in S_t.

We wish to point out another form of Lemma (2.1), helpful in other applications, which can be deduced from it by applying an affine transformation to $Q(D)$ as in [KRS].

Lemma (2.22). *Let* $z \in \mathbf{C}^n$, $\gamma \in \mathbf{C}$. *Let* $Q(D) = \Delta + z \cdot \nabla + \gamma$. *Let* $v \geq 0$, $v \in F_p$, $p > (n-1)/2$ *and* $f \in C_0^\infty(\mathbf{R}^n)$. *Then*

$$\int_{\mathbf{R}^n} |f|^2 v\, dx \leq C \int_{\mathbf{R}^n} |Q(D)f|^2 v^{-1}\, dx,$$

where C *does not depend on* f, γ *or* z.

3. Proof of Theorem (1.2)

Theorem (1.2) will be proved as usual using a Carleman inequality, but with a weighted $L^2(v^{-1})$ to $L^2(v)$ estimate in place of the unweighted (except for powers) $L^p - L^q$ estimates of [JK].

Lemma (3.1). *Suppose* $v \in F_p$ *for some* $p > (n-1)/2$. *For every* ε *with* $0 < \varepsilon < 2p - (n-1)$, *there are* $C, \beta > 0$ *depending only on* p, ε *and* n *such that*

$$\int_{\mathbf{R}^n} \frac{|u(x)|^2}{|x|^{2t_m+n\pm 2}} v(x)\, dx \leq C\|v\|_{F_p}^\beta \int_{\mathbf{R}^n} \frac{|\Delta u(x)|^2}{|x|^{2t_m+n\pm 2}} \frac{dx}{v(x)} \tag{3.2}$$

for all $u \in C_c^\infty(\mathbf{R}^n \backslash \{0\})$ *and all* $m \in \mathbf{Z}$, *where* $t_m = m - 1 + (1-\varepsilon)/2$ *if* $m \geq 0$ *and* $t_m = m - 1 + (1+\varepsilon)/2$ *if* $m \leq -1$, *and the* $\pm$ *sign is chosen according to the sign of* m: $-$ *if* $m \geq 0$ *and* $+$ *if* $m \leq -1$.

The case $t_m \to \infty$ in Lemma (3.1) will be used to prove part (A) of Theorem (1.3) while the case $t_m \to -\infty$ is used for part (B). We remark that the corresponding Carleman inequalities in [JK] have analogues for $t < 0$, but these do not seem to have appeared in the literature.

To prove the Carleman estimate (3.2), we use E. Stein's complex interpolation [ST_2] on an analytic family of operators, essentially $(-\Delta)^{z/2}$ minus a Taylor polynomial at the origin, as in [JK] but with two differences: we conjugate $(-\Delta)^{z/2}$ with a complex power of the weight $v(x)$ and in the case $t < 0$, we modify the kernel of $(-\Delta)^{z/2}$ so as to be C^∞ at infinity and then subtract a Taylor polynomial at infinity. We now describe the analytic family in detail. Let $\phi_z(x) = C_z|x|^{z-n}$ so that $\hat{\phi}_z(\xi) = (2\pi|\xi|)^{-z}$. Let $m \in \mathbf{Z}$ and $m - 1 < t < m$. In the case $m \geq 0$, define

$$S_z^t g(x) = \frac{v(x)^{z/4}}{\Gamma((n-z)/2)} \int_{\mathbf{R}^n} \left[\phi_z(x-y) - \sum_{l=0}^{m-1} \frac{1}{l!} \left(\frac{\partial}{\partial s}\right)^l \phi_z(sx - y)\bigg|_{s=0} \right] \times \left(\frac{|y|}{|x|}\right)^{t+(n-z)/2} v(y)^{z/4} g(y)\, dy$$

(where for $m = 0$, the sum is empty) and in the case $m \leq -1$, define

$$S_z^t g(x) = \frac{v(x)^{z/4}}{\Gamma((n-z)/2)} \int_{\mathbf{R}^n} |x|^{z-2} \times \left[\psi_z(x,y) - \sum_{l=0}^{|m|-2} \frac{1}{l!} \left(\frac{\partial}{\partial s}\right)^l \psi_z(s^{-1}x, y)\bigg|_{s=0} \right] \left(\frac{|y|}{|x|}\right)^{t+(n+z)/2} v(y)^{z/4} g(y)\, dy$$

where $\psi_z(x,y) = |x|^{2-z}\phi_z(x-y)$. Note that both $s \mapsto \phi_z(sx-y)$ and $s \mapsto \psi_z(s^{-1}x,y) = |s|^{n-2}|x|^{2-z}\phi_z(x-sy)$ are C^∞ near $s=0$ if $x,y \neq 0$—if n is odd, we restrict s to be nonnegative in $\psi_z(s^{-1}x,y)$ and interpret the symbol $f(s)|_{s=0}$ appearing above to mean $\lim_{s\to 0^+} f(s)$. Now the family of operators $\{S_z^t\}_{0\le \operatorname{Re} z \le n}$ is analytic in z and S_2^t satisfies the identity

$$\frac{u(x)v(x)^{1/2}}{|x|^{t\pm 1+n/2}} = S_2^t\left(\frac{\Delta u(y)}{v(y)^{1/2}|y|^{t\pm 1+n/2}}\right)(x), \qquad t \in \mathbf{R}\backslash\mathbf{Z},$$

whenever $u \in C_c^\infty(\mathbf{R}^n\backslash\{0\})$, and where $\pm$ is chosen according to the sign of m. Thus Lemma (3.1) is equivalent to the boundedness of S_2^t on $L^2(\mathbf{R}^n)$, and this in turn follows by complex interpolation [ST_2] once we have shown the boundedness of $S_{i\gamma}^t$ on $L^2(\mathbf{R}^n)$ (which for $t>0$ is Lemma 2.3 of [JK]) and, under the hypothesis $v \in F_p$, $p > (n-1)/2$, the boundedness of $S_{n-1+\varepsilon+i\gamma}^t$ on $L^2(\mathbf{R}^n)$, for $0 < \varepsilon/2 < p-(n-1)/2$, with norms in both cases dominated by $Ce^{c|\gamma|}$.

First we prove that

$$\int_{\mathbf{R}^n} |S_{i\gamma}^t f(x)|^2\,dx \le Ce^{c|\gamma|}\int_{\mathbf{R}^n} |f(x)|^2\,dx \tag{3.3}$$

for all $f \in L^2(\mathbf{R}^n)$, where C depends only on $\delta = \text{distance}(t,\mathbf{Z})$ and n. To this end we introduce the family of operators $\{T_z^t\}_{0\le \operatorname{Re} z\le 1}$ given by

$$T_z^t g(x) = \frac{1}{\Gamma((n-z)/2)}\int_{\mathbf{R}^n}\left[\phi_z(x-y) - \sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \phi_z(sx-y)\bigg|_{s=0}\right] \times \left(\frac{|y|}{|x|}\right)^t |y|^{-z} g(y)\,dy$$

for $m-1<t<m$, $m \in \mathbf{Z}$, $m \ge 0$ and by

$$T_z^t g(x) = \frac{1}{\Gamma((n-z)/2)}\int_{\mathbf{R}^n}\left[\psi_z(x,y) - \sum_{l=0}^{|m|-2}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \psi_z(s^{-1}x,y)\bigg|_{s=0}\right] \times \left(\frac{|y|}{|x|}\right)^{t+2-z} |y|^{-2} g(y)\,dy$$

for $m-1<t<m$, $m\in\mathbf{Z}$, $m \le 0$. Note that for $m=0$, the two definitions are identical and that for $m \ge 0$, T_z^t coincides with the family of operators T_z in [JK]. Clearly (3.3) is equivalent to

$$\int_{\mathbf{R}^n} (T_{i\gamma}^t g(x)|^2 \frac{dx}{|x|^n} \le Ce^{c|\gamma|}\int_{\mathbf{R}^n} |g(x)|^2\frac{dx}{|x|^n} \tag{3.4}$$

for all $g \in L^2(|x|^{-n}\,dx)$, where C depends only on δ and n. Since the case $m \ge 0$ of (3.4) is Lemma 2.3 of [JK], we suppose $m \le 0$. As in [JK] we prove

(3.4) by calculating the action of T_z^t on spherical harmonics for $0 < \operatorname{Re} z < 1$, computing the Mellin transform and then letting $\operatorname{Re} z \to 0^+$. The details are similar to those in [JK], so we will refer liberally to that paper for the sake of brevity.

Let $g(r\omega) = h(r)P_k(\omega)$, where $r > 0$, $\omega \in S^{n-1}$, $h \in C_c^\infty(\mathbf{R}_+ \backslash \{0\})$ and P_k is a spherical harmonic homogeneous of degree k. Then according to Lemma 3.2 of [JK], if $-1 < t < 0$ and $\operatorname{Re} z > 0$,

$$T_z^t g(r\omega) = \int_0^\infty L'\left(k, z, \frac{r}{s}\right)\left(\frac{r}{s}\right)^{z-t-2} h(s)\frac{ds}{s} P_k(\omega), \tag{3.5}$$

where

$$L'(k, z, u) = \begin{cases} d(\nu, z)u^{k+2-z}F(\nu - z/2, 1 - z/2; \nu; u^2), & 0 < u < 1, \\ d(\nu, z)u^{-n-k+2}F(\nu - z/2, 1 - z/2; \nu; u^{-2}), & 1 < u < \infty, \end{cases}$$

and

$$d(\nu, z) = \frac{2^{1-z}\Gamma(\nu - z/2)}{\Gamma(\nu)\Gamma(z/2)\Gamma(n/2 - z/2)},$$

$\nu = k + n/2$ and $F(a, b; c; x)$ denotes the hypergeometric function. (Note that $L'(k, z, u) = u^{2-z}L(k, z, u)$, where L is as on page 468 of [JK].) Now set $g_1(y) = |y|^{t-z}g(y)$ and $f_1(x) = |x|^{2-z}(-\Delta)^{-z/2}g_1(x)$ so that

$$T_z^t g(x) = \frac{1}{\Gamma(n/2 - z/2)}|x|^{-t+z-2}\left[f_1(x) - \sum_{l=0}^{|m|-2}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l f_1(s^{-1}x)\bigg|_{s=0}\right].$$

Using (3.5), it follows that

$$T_z^t g(r\omega) = \int_0^\infty L'_m\left(k, z, \frac{r}{s}\right)\left(\frac{r}{s}\right)^{z-t-2} h(s)\frac{ds}{s} P_k(\omega), \tag{3.6}$$

where

$$L'_m(k, z, u) = L'(k, z, u) - \sum_{l=0}^{|m|-2}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l L'(k, z, s^{-1}u)\bigg|_{s=0}.$$

Now let $R'_k(u) = L'_m(k, z, u)u^{-z-t-2}$. From the estimates for the hypergeometric function given in Remark 3.1 of [JK], it follows that $R'_k \in L^1(du/u)$ for $0 < \operatorname{Re} z < t - (m-1)$. Indeed, the only singularities of R'_k occur at $u = 0, 1$, and ∞. The singularity at $u = 1$ is integrable for $\operatorname{Re} z > 0$ as in [JK] while $|R'_k(u)| \le Cu^{-(|m|-2)}u^{\operatorname{Re} z-t-2}$ for $0 \le u \le \frac{1}{2}$ and $|R'_k(u)| \le Cu^{-(|m|-2)}u^{\operatorname{Re} z-t-2}$ for $2 \le u < \infty$. Thus R'_k is integrable at 0 and ∞ provided $-(|m|-2) + \operatorname{Re} z - t - 2 > 0$ and $-(|m|-1) + \operatorname{Re} z - t - 2 < 0$ for $m \le 0$, i.e., provided $m - 1 + \operatorname{Re} z < t < m + \operatorname{Re} z$, and in particular if $0 < \operatorname{Re} z < t - (m-1)$.

We now show that the formula for the Mellin transform of R_k given in [JK], $m \geq 0$, remains valid for R'_k, $m \leq 0$. According to (3.5) of [JK] (recall that $L' = u^{2-z}L$),

$$2^z \int_0^\infty L'(k,z,u)u^{-\lambda}\frac{du}{u} = 2^z \int_0^\infty L(k,z,u)u^{-(\lambda+z-2)}\frac{du}{u}$$
$$= \frac{\Gamma((k-\lambda-z+2)/2)\Gamma((n+k-2+\lambda)/2)}{\Gamma((k-\lambda+2)/2)\Gamma((n+k+\lambda+z-2)/2)\Gamma((n-z)/2)}$$

for $\operatorname{Re} z > 0$ and $-\operatorname{Re}(k+n-2) < \operatorname{Re}\lambda < k - \operatorname{Re} z + 2$. Now, in analogy with Lemma 3.4 of [JK], let $F(\lambda)$ denote 2^{-z} times the quotient of gamma functions above so that

$$F(\lambda) = \int_0^\infty L'(k,z,u)u^{-\lambda}\frac{du}{u} \quad \text{for } 0 < \operatorname{Re}\lambda < 2 - \operatorname{Re} z.$$

Suppose $L'(k,z,u) = \sum_{l=0}^\infty c_l u^{-l}$ for u large. Then, arguing as in [GS], for $m \leq -1$, $0 < \operatorname{Re} z < 1$, and $0 < \operatorname{Re}\lambda < 2 - \operatorname{Re} z$,

$$\begin{aligned}
F(\lambda) &= F(\lambda) - \int_1^\infty \sum_{l=0}^{|m|-2} c_l u^{-l-\lambda}\frac{du}{u} + \sum_{l=0}^{|m|-2}\frac{c_l}{l+\lambda} \\
&= \int_0^\infty L'(k,z,u)u^{-\lambda}\frac{du}{u} \\
&\quad - \int_1^\infty \left[\sum_{l=0}^{|m|-2}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l L'(k,z,s^{-1}u)\Bigg|_{s=0}\right] u^{-\lambda}\frac{du}{u} + \sum_{l=0}^{|m|-2}\frac{c_l}{l+\lambda} \\
&= \int_0^1 L'(k,z,u)u^{-\lambda}\frac{du}{u} + \int_1^\infty L'_m(k,z,u)u^{-\lambda}\frac{du}{u} + \sum_{l=0}^{|m|-2}\frac{c_l}{l+\lambda}.
\end{aligned}$$

However, the right-hand side above is meromorphic in $m+1 < \operatorname{Re}\lambda < 2 - \operatorname{Re} z$ and so must coincide with $F(\lambda)$ in that region. On the other hand, if $\operatorname{Re}\lambda < m+2$, then

$$\begin{aligned}
&\int_0^1 L'(k,z,u)u^{-\lambda}\frac{du}{u} + \sum_{l=0}^{|m|-2}\frac{c_l}{l+\lambda} \\
&\qquad = \int_0^1 L'(k,z,u)u^{-\lambda}\frac{du}{u} - \int_0^1 \sum_{l=0}^{|m|-2} c_l u^{-l-\lambda}\frac{du}{u} \\
&\qquad = \int_0^1 L'_m(k,z,u)u^{-\lambda}\frac{du}{u}
\end{aligned}$$

and combining these identities yields

$$F(\lambda) = \int_0^\infty L'_m(k,z,u)u^{-\lambda}\frac{du}{u}, \quad \text{for } m+1 < \operatorname{Re}\lambda < m+2.$$

Thus the Mellin transform of R_k' is given by

$$\begin{aligned}\widetilde{R}_k'(\eta) &= \int_0^\infty R_k'(u)u^{-i\eta}\frac{du}{u} = \int_0^\infty L_m'(k, z, u)u^{-(t+2-z+i\eta)}\frac{du}{u} \\ &= \frac{\Gamma((k-t-i\eta)/2)\Gamma((n+k+t-z+i\eta)/2)2^{-z}}{\Gamma((k-t+z-i\eta)/2)\Gamma((n+k+t+i\eta)/2)\Gamma((n-z)/2)}\end{aligned} \tag{3.7}$$

for $0 < \operatorname{Re} z < 1$ and $m - 1 + \operatorname{Re} z < t < m + \operatorname{Re} z$, $m \leq 0$.

We now complete the proof of (3.4) as in [JK] by expanding g in spherical harmonics as follows:

$$g(r\omega) = \sum_{\substack{k\geq 0\\ 1\leq l\leq a_k}} g_{k,l}(r)P_{k,l}(\omega),$$

where $g_{k,l} \in C_c^\infty(\mathbf{R}_+\backslash\{0\})$ and $\{P_{k,l}\}_{l=1}^{a_k}$ is an orthonormal basis for the vector space of spherical harmonics of degree k. If we set

$$T_{z,k}^t(h)(r) = \int_0^\infty R_k'\left(\frac{r}{s}\right)h(s)\frac{ds}{s},$$

then

$$T_{i\gamma}g(x) = \lim_{\varepsilon\to 0} T_{\varepsilon+i\gamma}g(x) = \lim_{\varepsilon\to 0}\sum_{k,l} T_{\varepsilon+i\gamma,k}(g_{k,l})(r)P_{k,l}(\omega)$$

and it follows from (3.6), (3.7) and Plancherel's theorem that

$$\begin{aligned}&\int_{\mathbf{R}^n}|T_{i\gamma}g(x)|^2\frac{dx}{|x|^n} \leq \lim_{\varepsilon\to 0}\sum_{k,l}\int_0^\infty |T_{\varepsilon+i\gamma,k}(g_{k,l})(r)|^2\frac{dr}{r} \\ &= \lim_{\varepsilon\to 0}\sum_{k,l}\int_{-\infty}^\infty\left|\frac{2^{-(\varepsilon+i\gamma)}\Gamma((k-t-i\eta)/2)\Gamma((n+k+t-\varepsilon-i\gamma+i\eta)/2)}{\Gamma((n-\varepsilon-i\gamma)/2)\Gamma((k-t+\varepsilon+i\gamma-i\eta)/2)\Gamma((n+k+t+i\eta)/2)}\right|^2|\tilde{g}_{k,l}(\eta)|^2\,d\eta.\end{aligned}$$

Using Stirling's formula as in [JK], the above is dominated by

$$\begin{aligned}Ce^{c|\gamma|}\sum_{k,l}\int_{-\infty}^\infty |\tilde{g}_{k,l}(\eta)|^2\,d\eta &= Ce^{c|\gamma|}\sum_{k,l}\int_0^\infty |g_{k,l}(r)|^2\frac{dr}{r} \\ &= Ce^{c|\gamma|}\int_{\mathbf{R}^n}|g(x)|^2\frac{dx}{|x|^n}.\end{aligned}$$

This completes the proof of (3.4) and (3.3).

To complete the proof of Lemma (3.1) it remains only to prove, under the hypothesis $v \in F_p$, $p > (n-1)/2$, the boundedness of the operator $S_{n-1+\varepsilon+i\gamma}^t$ on $L^2(\mathbf{R}^n)$, $0 < \varepsilon/2 < p - (n-1)/2$, with norm dominated by $Ce^{c|\gamma|}$. The key here is the following inequality of E. Stein [St$_1$] (cf. [Sa]):

$$\begin{aligned}&\left|\,|1-re^{i\theta}|^{-2\lambda} - \sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l |1-sre^{i\theta}|^{-2\lambda}\Big|_{s=0}\right| \\ &\leq C_\lambda r^{m-1+2\operatorname{Re}\lambda}|1-re^{i\theta}|^{-2\operatorname{Re}\lambda},\end{aligned} \tag{3.8}$$

for $r > 0$, $\theta \in \mathbf{R}$, $m \in \mathbf{Z}_+$ and $0 < \operatorname{Re}\lambda < \frac{1}{2}$ and where the constant C_λ is independent of m, r, θ and satisfies $C_\lambda \le C_\mu e^{c_\mu|\nu|}$, $\lambda = \mu + i\nu$. Fix $x, y \in \mathbf{R}^n$ momentarily and identify the points $0, y/|y|, x/|y|$ in $\mathbf{R}^n$ with the numbers $0, 1, re^{i\theta}$ in the complex plane. Since

$$\phi_z(sx - y) = c_z|y|^{z-n}\left|\frac{y}{|y|} - s\frac{x}{|y|}\right|^{z-n},$$

(3.8) becomes with $2\lambda = n - z$,

$$\tag{3.9} \begin{aligned} &\left|\left.\phi_z(x-y) - \sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \phi_z(sx-y)\right|_{s=0}\right| \\ &\quad \le C_z\left(\frac{|x|}{|y|}\right)^{m-1+n-\operatorname{Re} z}\phi_{\operatorname{Re} z}(x-y), \qquad 0 < \operatorname{Re}\frac{n-z}{2} < \frac{1}{2}. \end{aligned}$$

Now identify $0, x/|x|, y/|x|$ with $0, 1, re^{i\theta}$. Since

$$\psi_z(s^{-1}x, y) = c_z s^{n-2}|x|^{2-n}\left|\frac{x}{|x|} - s\frac{y}{|x|}\right|^{z-n},$$

(3.8) becomes with $2\lambda = n - z$,

$$\tag{3.10} \begin{aligned} &\left|\left.\psi_z(x,y) - \sum_{l=0}^{|m|-2}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \psi_z(s^{-1}x,y)\right|_{s=0}\right| \\ &\quad \le C_z\left(\frac{|y|}{|x|}\right)^{|m|-\operatorname{Re} z}\psi_z(x,y), \qquad 0 < \operatorname{Re}\frac{n-z}{2} < \frac{1}{2}. \end{aligned}$$

(Note that in obtaining (3.10), we use (3.8) with m replaced by $|m|-2-(n-2)$.) With $t_m = m - 1 + (1-\varepsilon)/2$ in case $m \ge 0$ and $t_m = m - 1 + (1+\varepsilon)/2$ in case $m \le -1$, we obtain from (3.9) and (3.10) that

$$|S^{t_m}_{n-1+\varepsilon+i\gamma}g(x)| \le C_\gamma v(x)^{(n-1+\varepsilon)/4}\int_{\mathbf{R}^n}\phi_{n-1+\varepsilon}(x-y)v(y)^{(n-1+\varepsilon)/4}|g(y)|\,dy$$

for $m \in \mathbf{Z}$ with C_γ independent of m. Thus the boundedness of $S^{t_m}_{n-1+\varepsilon+i\gamma}$ on L^2 with norm dominated by $Ce^{c|\gamma|}$ will follow from the inequality

$$\tag{3.11} \int_{\mathbf{R}^n}(I_{n-1+\varepsilon}f)^2 v^{(n-1+\varepsilon)/2} \le C\int_{\mathbf{R}^n} f^2 v^{-(n-1+\varepsilon)/2}, \qquad f \ge 0.$$

However, if $v \in F_p$, $p > (n-1+\varepsilon)/2$, then (3.11) holds with $C = C_{p,n,\varepsilon}$ by [CW], since we can replace v by $v^* = M(v^q)(x)^{1/q}$, $p > q > (n-1+\varepsilon)/2$, as in the proof of Lemma (2.2) in §2. Alternatively, (3.11) is equivalent to the trace inequality

$$\tag{3.12} \int_{\mathbf{R}^n}(I_{(n-1+\varepsilon)/2}f)^2 v^{(n-1+\varepsilon)/2} \le C\int_{\mathbf{R}^n} f^2, \qquad f \ge 0.$$

(To see this, simply compose (3.12) with its dual inequality,

$$\int (I_{(n-1+\varepsilon)/2}f)^2 \le C\int_{\mathbf{R}^n} f^2 v^{-(n-1+\varepsilon)/2},$$

and use $I_\alpha(I_\alpha f) = I_{2\alpha} f$.) See [F] and [KS] for the relevant results on (3.12). This completes the proof of the Carleman inequality, Lemma (3.1).

Proof of Theorem (1.2). (A) As in [JK], it suffices to show that if u vanishes to infinite order at a and satisfies the differential inequality $|\Delta u| \le v|u|$, then u vanishes identically in $B(a,R)$ whenever $\|\chi_{B(a,R)}v\|_{F_p} < \varepsilon(p,n)$. Here $B(a,R) = \{x: |x-a| < R\}$ if $a \in \mathbf{R}^n$ and $B(\infty, R) = \{x: |x| > R\}$ if $a = \infty$. Furthermore, we may assume $v \ge 1$ since $v+1$ also satisfies the hypotheses of Theorem (1.3). We begin with the case $a \in \mathbf{R}^n$ and suppose without loss of generality that $a = 0$. Momentarily fix $R > 0$ and let $\eta \in C_c^\infty(B(0,2R))$ satisfy $\eta = 1$ on $B(0,R)$. In addition, let $\psi \in C^\infty(\mathbf{R}^n)$ satisfy $\psi = 0$ on $B(0,1)$, $\psi = 1$ outside $B(0,2)$, $0 \le \psi \le 1$, and set $\psi_k(x) = \psi(kx)$ for $k \ge 1$. Then for $k \ge 4/R$ and $m \ge 0$, Lemma (3.1) gives

(3.13)

$$\begin{aligned}
&\int_{|x|<R} \frac{|\psi_k(x)u(x)|^2}{|x|^{2t_m+n-2}} v(x)\,dx \\
&\quad \le C_{p,n}\|\chi_{B(0,R)}v\|_{F_p}^\beta \int_{\mathbf{R}^n} \frac{|\Delta(\psi_k \eta u)(x)|^2}{|x|^{2t_m+n-2}} \frac{dx}{v(x)} \\
&\quad \le C_{p,n}\|\chi_{B(0,R)}v\|_{F_p}^\beta \left[\int_{\mathbf{R}^n} \frac{|\Delta\psi_k(x)|^2|u(x)|^2}{|x|^{2t_m+n-2}} \frac{dx}{v(x)} \right. \\
&\qquad\qquad + \int_{\mathbf{R}^n} \frac{|\nabla\psi_k(x)|^2|\nabla u(x)|^2}{|x|^{2t_m+n-2}} \frac{dx}{v(x)} \\
&\qquad\qquad \left. + \left(\int_{|x|<R} + \int_{|x|>R} \right) \frac{|\psi_k(x)|^2|\Delta(\eta u)(x)|^2}{|x|^{2t_m+n-2}} \frac{dx}{v(x)} \right] \\
&\quad = \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k + \mathrm{IV}_k .
\end{aligned}$$

If $\|\chi_{B(0,R)}v\|_{F_p} < \varepsilon(p,n)$, where $\varepsilon(p,n)$ is chosen so that $C_{p,n}\varepsilon(p,n)^\beta = \frac{1}{2}$, then III_k is at most one-half the left side of (3.13) and subtracting III_k from both sides of (3.13) we obtain

$$\int_{|x|<R} \frac{|\psi_k(x)|^2|u(x)|^2}{|x|^{2t_m+n-2}} v(x)\,dx \le 2(\mathrm{I}_k + \mathrm{II}_k + \mathrm{IV}_k).$$

However, letting $k \to \infty$ we get

$$\int_{|x|<R} \frac{|u(x)|^2}{|x|^{2t_m+n-2}} v(x)\,dx \le 2\int_{|x|>R} \frac{|\Delta(\eta u)(x)|^2}{|x|^{2t_m+n-2}} \frac{dx}{v(x)}, \qquad m \ge 0, \tag{3.14}$$

since $\lim_{k\to\infty} \mathrm{I}_k = \lim_{k\to\infty} \mathrm{II}_k = 0$ (whose proof we give in a moment). Now multiply both sides of (3.14) by R^{2t_m+n-2} and then let $m \to \infty$ to conclude that $|u|^2 v$, and hence u is identically zero on $B(0,R)$. It remains only to show $\lim_{k\to\infty} \mathrm{I}_k = \lim_{k\to\infty} \mathrm{II}_k = 0$. But

$$\mathrm{I}_k = \int_{1/k\le|x|\le 2/k} \frac{|\Delta\psi_k(x)|^2|u(x)|^2}{|x|^{2t_m+n-2}}\frac{dx}{v(x)} \le Ck^{2t_m+n+2}\int_{|x|\le 2/k}|u(x)|^2\,dx,$$

which tends to zero as $k\to\infty$ since u vanishes to infinite order at 0. Similarly

$$\begin{aligned}\mathrm{II}_k &\le Ck^{2t_m+n}\int_{|x|\le 2/k}|\nabla u(x)|^2\,dx\\ &\le Ck^{2t_m+n}\left[\left(\int_{|x|\le 4/k}|u(x)|^2\,dx\right)^{1/2}\left(\int_{|x|\le 4/k}|\Delta u(x)|^2\,dx\right)^{1/2}\right.\\ &\qquad\left. + k^2\int_{|x|\le 4/k}|u(x)|^2\,dx\right]\end{aligned}$$

by Lemma (1.6)(A) and this last term tends to zero as $k\to\infty$ since u vanishes to infinite order at 0 and $\Delta u \in L^2_{\mathrm{loc}}$.

Proof of Theorem (1.2). (B) In the case u vanishes to infinite order at infinity, we argue as in the proof of part (A) above, but with $\eta\in C^\infty$ such that $\eta = 1$ on $B(\infty, R) = \{x : |x| > R\}$ and $\eta = 0$ on $B(0, R/2)$; $\psi \in C^\infty$ such that $\psi = 0$ on $B(\infty, 1)$ and $\psi = 1$ on $B(0, \frac{1}{2})$; $\psi_k(x) = \psi(x/k)$. Furthermore, we may assume that $v \ge \varepsilon|x|^{-2}$ since $v + \varepsilon|x|^{-2}$ also satisfies the hypothesis of Theorem (1.2)(B). The result is that (3.14) becomes

$$\int_{|x|>R}\frac{|u(x)|^2}{|x|^{2t_m+n+2}}v(x)\,dx \le \int_{|x|<R}\frac{|\Delta(\eta u)(x)|^2}{|x|^{2t_m+n+2}}\frac{dx}{v(x)}, \qquad m\le -1.$$

Note that the hypothesis $\int_{\mathbf{R}^n}|\Delta u|^2 < \infty$ is needed to show that the term corresponding to II_k in (3.13) tends to zero as k tends to infinity. Now multiply both sides by R^{2t_m+n+2} and then let $m\to-\infty$ to conclude that $u\equiv 0$ in $\{x : |x| > R\}$. This completes the proof of Theorem (1.2).

4. Unique continuation in low dimensions

We begin by proving Theorem (1.4). Recall that by the definition of $|||v|||$, we have

$$\int_{\mathbf{R}^n}|I_1 f(x)|^2 v(x)\,dx \le |||v|||\int_{\mathbf{R}^n}|f(x)|^2\,dx, \quad \text{for } f\ge 0,$$

where I_1 denotes the fractional integral of order one. If we compose this inequality with its dual, $\int |I_1 f|^2 \le |||v|||\int |f|^2 v^{-1}$, we obtain the equivalent inequality (for the reverse implication, apply Hölder's inequality to $\int |I_1 f|^2 = \int f I_2 f$)

$$\text{(4.1)}\qquad \int_{\mathbf{R}^n}|I_2 f(x)|^2 v(x)\,dx \le |||v|||^2\int_{\mathbf{R}^n}|f(x)|^2\frac{dx}{v(x)}, \quad \text{for } f\ge 0.$$

We now prove Theorem (1.4) by using the following pointwise inequality [Sa], which essentially corresponds to the cases $\lambda = 0$ and $\frac{1}{2}$ of (3.8):

$$(4.2)\qquad \left|\phi_2(x-y) - \sum_{l=0}^{m-1} \frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \phi_2(sx-y)\Big|_{s=0}\right| \le C\left(\frac{|x|}{|y|}\right)^m \phi_2(x-y),$$

for all $m \in \mathbf{Z}_+$, $x, y \in \mathbf{R}^n$, $n = 2$ or 3, and where C is independent of m, x and y. If $n = 2$, we must restrict $|x|, |y| < \frac{1}{4}$.

Now suppose u vanishes to infinite order at 0 and satisfies $|\Delta u| \le v|u|$. With η and ψ_k as in §3, we have for $m \in \mathbf{Z}_+$,

$$\begin{aligned}
&\int_{|x|<R} \frac{|\psi_k(x)u(x)|^2}{|x|^{2m}} v(x)\,dx \\
&= \int_{|x|<R} |x|^{-2m} \left|\int \phi_2(x-y)\Delta(\psi_k \eta u)(y)\,dy\right|^2 v(x)\,dx \\
&= \int_{|x|<R} |x|^{-2m} \\
&\quad \times \left|\int \left[\phi_2(x-y) - \sum_{l=0}^{m-1} \frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \phi_2(sx-y)\Big|_{s=0}\right] \Delta(\psi_k \eta u)(y)\,dy\right|^2 v(x)\,dx \\
&\le C^2 \int_{|x|<R} \left(\int \phi_2(x-y) \frac{|\Delta(\psi_k \eta u)(y)|}{|y|^m}\,dy\right)^2 v(x)\,dx \quad \text{by (4.2)} \\
&\le C^2 |||\chi_{B(0,R)} v|||^2 \int_{\mathbf{R}^n} \frac{|\Delta(\psi_k \eta u)(x)|^2}{|x|^{2m}} \frac{dx}{v(x)} \quad \text{by (4.1)}.
\end{aligned}$$

Provided R is small enough that $C^2 |||\chi_{B(0,R)} v|||^2 < \frac{1}{2}$, we can now argue as in the proof of Theorem (1.3) and conclude that $u \equiv 0$ on $B(0,R)$. A standard connectedness argument then completes the proof of Theorem (1.4).

We now restrict attention to dimension $n = 2$ and give the proof of Theorem (1.5). In order to handle the gradient term in the differential inequality $|\Delta u| \le w|\nabla u|$, we use a pointwise inequality corresponding to (4.2) but with ϕ_2 replaced by $R_j \phi_1$, where R_j denotes the jth Riesz transform, $j = 1, 2$:

$$(4.3)\qquad \left|R_j\phi_1(x-y) - \sum_{l=0}^{m-1} \frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l R_j\phi_1(sx-y)\Big|_{s=0}\right| \le \left(\frac{|x|}{|y|}\right)^m \phi_1(x-y),$$

for all $m \in \mathbf{Z}_+$ and x, y in $\mathbf{R}^2$. The proof of (4.3) is easy due to the especially simple form of $R_j \phi_1$;

$$(R_j\phi_1)\hat{\ }(\xi) = \frac{\xi_j}{|\xi|}\hat{\phi}_1(\xi) = \frac{\xi_j}{|\xi|^2}$$

implies $R_j\phi_1(x) = cx_j/|x|^2$ $(n = 2) = c\langle x, e_j\rangle/|x|^2$, a rational function of x.

To prove (4.3), we introduce the function

$$\begin{aligned}\psi_\lambda(z) &= \frac{e^{i\lambda}\mathrm{Re}(e^{-i\lambda}(1-z))}{|1-z|^2} = \frac{(1-z)+e^{i2\lambda}(1-\overline{z})}{2|1-z|^2} \\ &= \frac{1}{2(1-\overline{z})} + \frac{e^{i2\lambda}}{2(1-z)}, \qquad z \in \mathbf{C}.\end{aligned}$$

Then

$$\begin{aligned}&\left|\psi_\lambda(z) - \sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \psi_\lambda(sz)\Big|_{s=0}\right| = \left|\psi_\lambda(z) - \sum_{l=0}^{m-1}\frac{\overline{z}^l + e^{i2\lambda}z^l}{2}\right| \\ &\quad = \left|\frac{\overline{z}^m}{2(1-\overline{z})} + e^{i2\lambda}\frac{z^m}{2(1-z)}\right| \le |z|^m|1-z|^{-1},\end{aligned} \tag{4.4}$$

for all $m \in \mathbf{Z}_+$. Now momentarily fix x, y in $\mathbf{R}^2$ and, by rotating the plane, identify $y/|y|$ and $x/|y|$ with 1 and z in the complex plane. Suppose that under this rotation, the unit vector e_j in $\mathbf{R}^2$ is identified with $e^{i\lambda}$ in $\mathbf{C}$. Then

$$R_j\phi_1(x-y) = -|y|^{-1}R_j\phi_1\left(\frac{y}{|y|} - \frac{x}{|y|}\right) = -|y|^{-1}e^{-i\lambda}\psi_\lambda(z)$$

and (4.4) yields

$$\begin{aligned}&\left|R_j\phi_1(x-y) - \sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l R_j\phi_1(sx-y)\Big|_{s=0}\right| \le |y|^{-1}|z|^m|1-z|^{-1} \\ &\quad = \left(\frac{|x|}{|y|}\right)^m |y|^{-1}\left|\frac{y}{|y|} - \frac{x}{|y|}\right|^{-1} = \left(\frac{|x|}{|y|}\right)^m |x-y|^{-1},\end{aligned}$$

which is (4.3).

Now suppose u vanishes to infinite order at 0 and satisfies the differential inequality $|\Delta u| \le w|\nabla u|$. Let η and ψ_k be as in §3. We use the following weighted inequality (which follows from [CW] as in the proof of Lemma 1.2 in §2 above)

$$\int (I_1 g)^2 w \le C_p\|w^2\|_{F_{p/2}}\int g^2 w^{-1}, \quad \text{for } g \ge 0, \tag{4.5}$$

for $p > 1$, $n = 2$, together with the identity

$$\begin{aligned}|\nabla f(x)|^2 &= \sum_{j=1}^{2}\left|\int_{\mathbf{R}^n} R_j\phi_1(x-y)\Delta f(y)\,dy\right|^2 \\ &= \sum_{j=1}^{2}\left|\int_{\mathbf{R}^n}\left[R_j\phi_1(x-y) - \sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l R_j\phi_1(sx-y)\Big|_{s=0}\right]\Delta f(y)\,dy\right|^2\end{aligned} \tag{4.6}$$

for $f \in C_c^\infty(\mathbf{R}^2\backslash\{0\})$, to obtain

(4.7)

$$\begin{aligned}
&\int_{|x|<R} \frac{|\nabla(\psi_k u)(x)|^2}{|x|^{2m}} w(x)\,dx \\
&= \sum_{j=1}^{2} \int_{|x|<R} |x|^{-2m} \left| \int_{\mathbf{R}^n} \left[R_j\phi_1(x-y) - \sum_{l=0}^{m-1} \frac{1}{l!}\left(\frac{\partial}{\partial s}\right)^l \right.\right. \\
&\qquad\qquad \left.\left. \times R_j\phi_1(sx-y)\Big|_{s=0} \right] \Delta(\psi_k \eta u)(y)\,dy \right|^2 w(x)\,dx, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (4.6)}, \\
&\le 2\int_{|x|<R} \left(\int_{\mathbf{R}^n} \phi_1(x-y) \frac{|\Delta(\psi_k \eta u)(y)|}{|y|^m}\,dy \right)^2 w(x)\,dx, \quad \text{by (4.3)}, \\
&\le C_p \|\chi_{B(0,R)} w^2\|_{F_{p/2}}^{1/2} \int_{\mathbf{R}^n} \frac{|\Delta(\psi_k \eta u)(x)|^2}{|x|^{2m}} \frac{dx}{w(x)}, \quad \text{by (4.5)}, \\
&\le C_p \|\chi_{B(0,R)} w^2\|_{F_{p/2}}^{1/2} \left[\int \frac{|\Delta\psi_k|^2 |u|^2}{|x|^{2m}} \frac{dx}{w(x)} + \int \frac{|\nabla\psi_k|^2 |\nabla u|^2}{|x|^{2m}} \frac{dx}{w(x)} \right. \\
&\qquad\qquad \left. + \left(\int_{|x|<R} + \int_{|x|>R} \right) \frac{|\psi_k|^2 |\Delta(\eta u)|^2}{|x|^{2m}} \frac{dx}{w(x)} \right] \\
&= \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k + \mathrm{IV}_k .
\end{aligned}$$

Now I_k, II_k, and IV_k can be handled just as in the proof (see (3.13)) of Theorem (1.3). From the differential inequality $|\Delta u| \le w|\nabla u|$, we obtain

$$\begin{aligned}
\mathrm{III}_k &= C_p \|\chi_{B(0,R)} w^2\|_{F_{p/2}}^{1/2} \int_{|x|<R} \frac{|\psi_k|^2 |\Delta u|^2}{|x|^{2m}} \frac{dx}{w(x)} \\
&\le CC_p \|\chi_{B(0,R)} w^2\|_{F_{p/2}}^{1/2} \int_{|x|<R} \frac{|\psi_k|^2 |\nabla u|^2}{|x|^{2m}} w(x)\,dx \\
&\le CC_p \|\chi_{B(0,R)} w^2\|_{F_{p/2}}^{1/2} \left[\int_{|x|<R} \frac{|\nabla(\psi_k u)|^2}{|x|^{2m}} w(x)\,dx \right. \\
&\qquad\qquad \left. + \int_{|x|<R} \frac{|\nabla\psi_k|^2 |u|^2}{|x|^{2m}} w(x)\,dx \right] \\
&= \mathrm{V}_k + \mathrm{VI}_k ,
\end{aligned}$$

since $\psi_k \nabla u = \nabla(\psi_k u) - (\nabla\psi_k)u$. Now if R is so small that $CC_p\|\chi_{B(0,R)} w^2\|_{F_{p/2}}^{1/2} \le \frac{1}{2}$, then V_k is at most one-half the left side of (4.7) and so can be subtracted from both sides as in the proof (see (3.13)) of Theorem (1.3). It remains only to show that $\lim_{k\to\infty} \mathrm{VI}_k = 0$. However, since $\nabla\psi_k$ is supported in $\{x: 1/k \leq |x| \leq 2/k\}$,

$$\begin{aligned}\mathrm{VI}_k &\leq Ck^{2m+2}\int_{|x|\leq 2/k} |u(x)|^2 w(x)\,dx \\ &\leq Ck^{2m+2}\left[\int_{|x|\leq 2/k}\left|u(x) - \frac{1}{B(0,2/k)}\int_{B(0,2/k)} u\right|^2 w(x)\,dx \right.\\ &\qquad \left. + \left(\int_{B(0,2/k)} w\right)\left(\frac{1}{B(0,2/k)}\int_{B(0,2/k)} |u|\right)^2\right] \\ &= \mathrm{VII}_k + \mathrm{VIII}_k .\end{aligned}$$

Now $\lim_{k\to\infty} \mathrm{VIII}_k = 0$ since u vanishes to infinite order at 0 and

$$\begin{aligned}\mathrm{VII}_k &\leq Ck^{2m+2}\int_{|x|\leq 2/k} \left|I_1\left[\chi_{B(0,2/k)}(y)|\nabla u(y)|\right](x)\right|^2 w(x)\,dx \\ &\leq Ck^{2m+2}\int_{|x|\leq 2/k} |\nabla u(x)|^2 \frac{dx}{w(x)}, \quad \text{by (4.5)}.\end{aligned}$$

This last expression tends to zero as k tends to infinity by the argument used to handle term II_k in the proof of Theorem (1.3) in §3. This completes the proof of Theorem (1.5).

We remark that Theorem (1.5) can be sharpened in the spirit of Theorem (1.4) by replacing the quantity $\|w^2\|_{F_{p/2}}$, $p > 1$, in (4.5) with the norm of the embedding $I_{1/2}: L^2 \to L^2(w)$.

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Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1 Canada